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Contact problems and depth-sensing nanoindentation for frictionless and frictional boundary conditions

Feodor M. Borodich *, Leon M. Keer

Theoretical and Applied Mechanics, Technological Institute, Northwestern University, 2145 Sheridan Road, Evanston, IL 60208-3109, USA

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Abstract

The Hertz type contact problems and the analytical treatment of depth-sensing nanoindentation are under consideration. Fundamental relations of nanoindentation tests are derived for various boundary conditions within the contact region. First, a frictionless contact problem for a convex punch of revolution is studied and the connection between Galin and Sneddon solutions is shown. The Bulychev–Alekhin–Shorshorov (BASh) relation that is commonly used for evaluation of elastic modulus of materials by nanoindentation, is discussed. An analogous relation is derived that is valid for adhesive (no-slip) contact. Similarly to the Pharr–Oliver–Brotzen frictionless analysis, the obtained relation is independent of the geometry of the punch. Further, we study solutions to adhesive contact for punches whose shapes are described by monomial functions and obtain exact solutions for punches of arbitrary degrees of the monoms. These formulae are similar to the formulae of the frictionless Galin solutions and coincide with them when the material is incompressible. Finally, indenters having some deviation from their nominal shapes are considered. It is argued that for shallow indentation where the tip bluntness is on the same order as the indentation depth, the indenter shapes can be well approximated by non-axisymmetric monomial functions of radius. In this case problems obey the self-similar laws. Using one of the authors' similarity approach to three-dimensional contact problems and the corresponding formulae, other fundamental relations are derived for depth of indentation, size of the contact region, load, hardness, and contact area, which are valid for both elastic and non-elastic, isotropic and anisotropic materials for various boundary conditions. In particular, it is shown that independently of the boundary conditions, the current area of the contact region is a power-law function of the current depth of indentation whose exponent is equal to a half of the degree of the monomial function of the shape.

Keywords: Adhesive contact; No-slip; Nano-indentation; Elastic modulus; Hardness

* Corresponding author. Present address: School of Engineering, Cardiff University, P.O. Box 925, Queen's Buildings, Cardiff CF24 3AT, UK. Tel.: +1-847-4918439; fax: +1-847-4914011.

E-mail address: f-borodich@northwestern.edu (F.M. Borodich).

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1. Introduction

Indentation testing is widely used for analysis and estimations of mechanical properties of materials. Historically, first indentation tests were developed for hardness measurements. Then they were used for extracting the mechanical properties of materials. The estimations of the thin film mechanical properties can be affected by various factors (see, e.g. a discussion by Borodich et al., 2003). In this paper we study connections between Hertz type contact problems and nanoindentation tests and in particular, the influence of boundary conditions on analytical treatment of nanoindentation tests.

1.1. Hardness measurements

The idea of the hardness measurement traces back to Réaumur (1722) (see, e.g. review of Williams (1942)), who suggested to compare relative hardness of two contacting materials. However, the analytical approach to the problem goes back to Hertz. In January 1881 Hertz submitted his famous paper on contact theory to the Journal reine und angewandte Mathematik. The paper was published in 1882 (Hertz, 1882a). The same year he published another paper on contact problems where he suggested a way to evaluate hardness of materials. To be more specific we will cite him. He wrote: "*The hardness of a body is to be measured by the normal pressure per unit area which must act at the centre of a circular surface of pressure in order that in some point of the body the stress may just reach the limit consistent with perfect elasticity*" (Hertz, 1882b). His contact theory (Hertz, 1882a) is of a great practical importance and is used in a number models of contact (see, e.g. a discussion by Johnson (1982)). However, his above suggestion to measure the hardness of a material by the initiation of plastic yield under an impressed hard ball (Hertz, 1882b), was found to be impracticable Johnson (1985). Indeed, as early as in 1909 it was showed by Dinnik (1952) for a circular contact region and later by Belyaev (see §28 by Belyaev (1924)) for an elliptic contact region that according to Hertz contact theory, the point of maximum shearing stresses and consequently the point of first yield is beneath the contact surface and it is normally hidden from view (Fischer-Cripps, 1997). Hence, in spite of existence of some experimental techniques which give possibilities to detect plastic region below the surface (see e.g., Fischer-Cripps, 1997), it is rather difficult to detect the first yield point experimentally.

Since that time, various experimental techniques were developed for hardness measurements by indentation and various definitions of hardness were also introduced. Brinell (1900) delivered a lecture where he described existing experimental means for hardness measurements and presented another simple test (Brinell test) based on indentation of hard balls. Brinell assumed the test could give a single numerical expression that may be used as a hardness number. However, soon after this Meyer (1908) showed that the hardness of a metal cannot truly be represented by a single number and $P = ka^n$ where P is the load, k is an empirical coefficient, n is an exponent, and a is the radius of the impression after unloading.

The hardness H was defined originally as the ratio of the maximum indentation force to the area of the imprint after unloading

$$\text{Hardness} = \frac{\text{Load}}{\text{Area of imprint}}.$$

Brinell considered the area of curved surface, and the Brinnell hardness is usually defined as

$$H_{\mathrm{B}} = \frac{P}{A}, \quad A = \frac{\pi D}{2}(D - \sqrt{D^2 - 4a^2}),$$

where D is the diameter of the ball, while Meyer suggested using the area of the impression projected on the initial contact plane. Hence, the Meyer hardness is defined as (see, e.g. Tabor, 1951)

$$H_{\mathrm{M}} = \frac{P}{A}, \quad A = \pi a^2.$$

Thus, the Hertz linearized formulation of a boundary value problem may be applied to the Mayer approach, while it is not applicable to the Brinell test. A semi-analytical treatment of the Meyer test was given by Tabor (1951). Another treatment of the Meyer test based on the similarity approach, was given by Borodich (1989, 1993).

However, hardness is now often defined as the ratio of the maximum indentation force to the contact area or as the ratio of current contact force to the current contact area

$$\text{Hardness} = \frac{\text{Load}}{\text{Area of contact}}.$$

For example, Bhattacharya and Nix (1988) defined the hardness as the load divided by projected area under the indenter at various points on the loading curve. In this paper, we will adopt this definition.

Compared with spherical indenters, conical and pyramidal indenters have the advantage that geometrically similar impressions are obtained at different loads even in the non-linearized formulation (Smith and Sandland, 1925; Mott, 1956). Apparently, Ludwik (1908) was the first to use a diamond cone in a hardness test. In 1922 two other very popular indenters were introduced. Rockwell (1922) introduced a sphero-conical indenter (the Rockwell indenter), while Smith and Sandland (1922, 1925) suggested using a square-base diamond pyramid (the Vickers indenter). These and other classic methods of measuring hardness are described in details by Williams (1942), Mott (1956), and also in various standard textbooks. However, there is a difficulty in machining a four-sided indenter in such a way that the sides meet in a point and not as a chisel edge. This is why Berkovich and his research colleagues suggested the three-sided indenters for micro-hardness tests (Khrushchov and Berkovich, 1950; Mott, 1956).

1.2. Depth-sensing techniques

The further progress in micro- and nano-hardness tests is mainly due to introduction of depth-sensing indentation, i.e. the continuously monitoring the displacement of the indenter into the sample surface for both loading and unloading branches. According to Bulychev and Alekhin (1990), the idea of the continuous monitoring the displacement of the indenter was first introduced by Grodzinski (1953). However, the modern depth-sensing indentation technique, based on the use of electronics, was first introduced by Kalei (1968), who recorded load–depth diagrams for various metals and minerals. For example, the diagram was recorded for a chromium film of 1 μm thickness when the maximum depth of indentation was 150 nm. This revolutionary technique was developed very rapidly, first in the former Soviet Union (see, e.g., Alekhin et al., 1972; Ternovskii et al., 1973; Grigor'ev et al., 1977) and then world-wide. Pethica et al. (1983) reported that they monitored indentations to depths as low as 20 nm. Modern sensors can accurately monitor the load and the depth of indentation in the micro-Newton and few nanometer scale, respectively.

Introduction of a method of determination of Young's modulus according to the indentation diagram (Fig. 1) was a very important step in interpretation of indentation tests. The method was introduced by Bulychev and co-workers (Bulychev et al., 1975, 1976; Shorshorov et al., 1981) in 1975. Evidently, the load–displacement diagram at loading reflects both elastic and plastic deformations of the material, while the unloading is taking place elastically. The boundary demarcating the elastic and plastic regions may only be estimated by numerical techniques, for example by the finite element method. Therefore, Bulychev et al. (1975) applied the elastic contact solution to unloading path of the load–displacement diagram assuming the non-homogeneity of the residual stress field in a sample after plastic deformation may be neglected. The Bulychev–Alekhin–Shorshorov (BASh) equation for the stiffness S of the upper portion of the load–displacement curve at unloading is the following:

$$S = \frac{dP}{dh} = \frac{2\sqrt{A}}{\sqrt{\pi}} E^*, \qquad (1)$$

where $A = \pi a^2$.

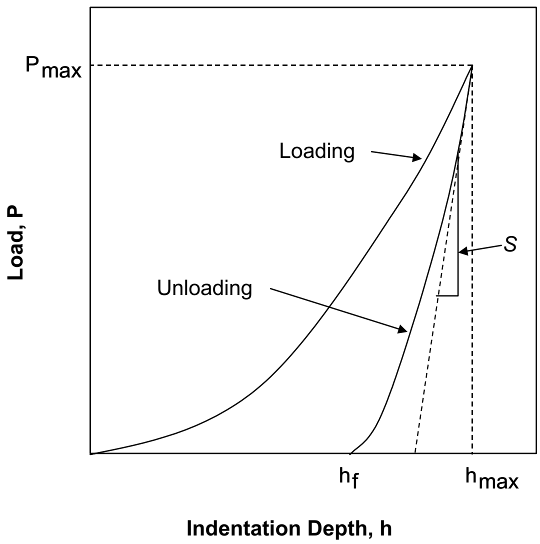

Fig. 1. A sketch of the load–indentation depth curve.

Here P is the external load, h is the indentation depth of the indenter tip, A is the contact area, a is the radius of the contact region, and E^* is the reduced Young's modulus. This modulus can be obtained from the following formula

$$\frac{1}{E^*} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}.$$

Here E_i and ν_i $(i = 1, 2)$ are the Young's modulus and the Poisson ratio of the first and the second solid respectively. If the indenter is rigid, i.e. $E_2 = \infty$ then $E^* = E/(1-\nu^2)$ where $E = E_1$ and $\nu = \nu_1$ are the Young's modulus and the Poisson ratio of the half-space, respectively.

The BASh relation is an example of fundamental relations which can be obtained from the analysis of frictionless contact problems (see, e.g. Hay et al., 1999). Note that the BASh relation is valid only for frictionless elastic contact, while the problems of adhesive contact, i.e. when there is no relative slip between the surfaces of contacting solids, are more complicated. The latter problems were considered by various authors (see, e.g. Mossakovskii, 1954, 1963; Goodman, 1962; Keer, 1967; Spence, 1968; Khadem and Keer, 1974 and the literature therein). However, the main results were obtained by Mossakovskii (1954, 1963) and Spence (1968). A detailed discussion of various methods of solving adhesive contact problems was performed by Gladwell (1980). Recently, the authors developed the Mossakovskii approach to the adhesive problem and derived a relation that is analogous to the BASh relation (Borodich and Keer, 2004).

It is well known (Borodich and Galanov, 2002) that the main feature of the Hertz type contact problems and the main difficulty of solving these problems is that the contact region is not known a priori. Hence, even if the contact problem for linear elastic solids is considered, the problem is non-linear. However, the classical non-linear Hertzian contact problem is self-similar. Hence, the non-linear 3D Hertz type contact problem for linear elastic materials can be considered as steady-state. The self-similarity of the 3D problems for isotropic elastic solids was discovered independently by Galanov (1981a) and Borodich (1983). Starting with the pioneering work by Galanov (1981b), the similarity approach has been applied for numerical simulations of the hardness probe by sharp indentors (Galanov, 1982; Galanov and Grigor'ev, 1986, see also recent papers by Larsson, 2001 and Mata et al., 2002). It was shown by Borodich (1988, 1989, 1993) that the self-similarity approach is valid for non-linear anisotropic materials in both frictionless and frictional cases. See also a recent discussion by Borodich and Galanov (2002). Roughly speaking, if the stress–strain relation of the coating is $\sigma \sim \epsilon^{\kappa}$ where κ is the work-hardening exponent of the constitutive relationship and the indenter shape is described by $f \equiv H_d$ then the problem is self-similar, where H_d is a homogeneous function. Following the recent approach by Borodich et al. (2003), we will apply the similarity approach to blunted indenters having some deviation from their nominal shapes.

The paper is organized as follows: in Section 2 we consider frictionless Hertz type contact problems and their applications for interpretation of micro- and nano-indentation tests. Although the formulation of the Hertz type problem is given for both linear and non-linear materials, further in this paragraph we discuss only contact for linear elastic materials and show that the BASh relation can be derived from the Galin solution for frictionless axi-symmetric contact. In Section 3 we consider adhesive (no-slip) axi-symmetric contact problems for linear elastic materials and derive a relation for the slope of the load–displacement curve that is similar to the BASh relation. However, the final formula differs by a factor that depends on the Poisson ratio of the material. We obtain also exact solutions for monomial punches of revolution and show that in the case of incompressible materials the derived solutions are identical with the Galin solutions obtained for frictionless contact. In Section 4 we consider indenters having some deviation from their nominal shapes. We argue that for both axi-symmetric and three-dimensional cases (3D), the indenter shape near the tip may be well approximated by monomial functions of radius. Since Hertz type contact problems for such indenters are self-similar (Borodich, 1989, 1993), other fundamental relations are derived for depth of indentation, size of the contact region, load, hardness, and contact area. Contrary to the relations obtained in previous paragraphs, these relations are valid for both elastic and non-elastic (under progressive loading), isotropic and anisotropic materials provided that the stress–strain curves are of power law type. These relations are especially important for shallow indentation, where the tip bluntness is on the same order as the indentation depth. The relations depend on the material hardening exponent and the degree of the monomial function of the shape. Finally, we discuss uncertainties in nanoindentation measurements that arise from geometric deviation of the indenter tip from its nominal geometry. We argue that some of the uncertainties can be explained and quantitatively described using our new relations. In Appendix A we show the equivalence of the Galin (1946) and the Sneddon (1965) solutions. In Appendix B we derive the BASh relation using the Sneddon (1965) solution. In Appendix C we discuss the two-dimensional (2D) Abramov–Muskhelishvili problem of adhesive contact for a flat punch.

2. Frictionless indentation

2.1. Hertz type contact problems for rigid indenters

Hertz (1882a,b) considered three-dimensional (3D) frictionless contact of two isotropic, linear elastic solids. It is possible to show that the problem is mathematically equivalent to the problem of contact between an indenter whose shape function f is equal to the initial distance between the surfaces, i.e. $f = f_1 + f_2$ where f_1 and f_2 are the shape functions of the solids, and a half-space. In turn, this problem can be reduced to the problem of contact between a rigid indenter (a punch) and an elastic half-space. Let us consider the Hertz type contact problems for rigid indenters. It is assumed that a rigid indenter (a punch) is pressed by the force P to a boundary of the contacting solid. In a geometrically linear formulation of the contact problem, this solid can be considered as a positive half-space $x_3 \geqslant 0$. Initially, there is only one point of contact between the punch and the half-space. Let us put the origin (O) of Cartesian x_1, x_2, x_3 coordinates at the point of initial contact between the punch and the half-space $x_3 \geqslant 0$. We denote the boundary plane $x_3 = 0$ by $\mathbb{R}^2$. Hence, the equation of the surface given by a function f, can be written as $x_3 = -f(x_1, x_2)$, $f \geqslant 0$.

After the punch contacts with the half-space, displacements u_i and stresses σ_{ij} are generated. If material properties are time independent then the current state of the contact process can be completely characterized by an external parameter ($\mathscr{P}$), e.g., the compressing force (P), the relative approach of the bodies (h) (for a rigid indenter, h is the depth of indentation) and the size of the contact region (l). For axi-symmetric problems l is equal to the contact radius a. It should be noted that (l) can be used as the external parameter of the problem only for convex punches (Borodich and Galanov, 2002).

Thus, it is supposed that the shape of the punch and the external parameter of the problem $\mathscr{P}$ are given and one has to find the bounded region G on the boundary plane $x_3 = 0$ of the half-space at the points where the punch and the medium are in mutual contact, displacements u_i, and stresses σ_{ij}. If the pressing force P is taken as the external parameter $\mathscr{P}$ then one has to find the depth of indentation h and the size of the contact region l. If h is taken as $\mathscr{P}$ then one has to find P and l.

2.2. Formulation of a Hertz type contact problem

In the general case of a 3D Hertz type contact problem, it is not assumed that the punch shape is described by an elliptic paraboloid and the contact region is an ellipse as Hertz did, but the problem formulation has the same main features as the original Hertz problem (Hertz, 1882a). Hence, the formulation of the problem is geometrically linear, the contact region is unknown and should be found, only vertical displacements of the boundary are taken into account, and the problem has the same boundary conditions within and outside the contact region as in the original Hertz problem.

In the problem the quantities sought satisfy the following equations

$$\begin{aligned} &\sigma_{ji,j} = 0, \quad i,j = 1,2,3; \\ &\sigma_{ij} = \mathscr{F}(\epsilon_{ij}), \quad \epsilon_{ij} = (u_{i,j} + u_{j,i})/2; \\ &\iint_{\mathbb{R}^2} \sigma_{33}(\mathbf{x}, \mathscr{P})\,d\mathbf{x} = -P, \end{aligned} \tag{2}$$

in which ϵ_{ij} are the components of the strain tensor and $\mathscr{F}$ is the operator of constitutive relations for the material. The material behavior of the medium may be linear and non-linear, anisotropic or isotropic, depending on the form of the operator $\mathscr{F}$. For anisotropic, linear elastic media, the constitutive relations have the form of Hooke's law

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl} \quad \text{or} \quad \sigma_{ij} = c_{ijkl}u_{k,l}, \quad c_{ijkl} = c_{jikl} = c_{klij},$$

where c_{ijkl} are components of the tensor of elastic constants. In particular, in the case of an isotropic elastic medium, Hooke's law becomes

$$\sigma_{ij} = \lambda\delta_{ij}u_{k,k} + \mu(u_{i,j} + u_{j,i}),$$

where λ and μ are the Lamé coefficients, and δ_{ij} is the Kronecker delta. Here and henceforth, a comma before the subscript denotes the derivative with respect to the corresponding coordinate; and summation from 1 to 3 is assumed over repeated Latin subscripts.

The displacement vector $\mathbf{u}$ should satisfy the conditions at infinity

$$\mathbf{u}(\mathbf{x}) \to 0 \quad \text{when } |\mathbf{x}| \to \infty. \tag{3}$$

Let us define the contact region G as an open region such that if $\mathbf{x} \in G$ then the gap $(u_3 - g)$ between the punch and the half-space is equal to zero and surface stresses are non-positive, while for $\mathbf{x} \in \mathbb{R}^2 \setminus G$ the gap is positive and the stresses are equal to zero. Thus, $\mathbf{u}$ and σ_{ij} should satisfy the following boundary conditions within and outside the contact region

$$u_3(\mathbf{x};\mathscr{P}) = g(\mathbf{x};\mathscr{P}), \quad \sigma_{33}(\mathbf{x};\mathscr{P}) \leqslant 0, \quad \mathbf{x} \in G(\mathscr{P}),$$
$$u_3(\mathbf{x};\mathscr{P}) > g(\mathbf{x};\mathscr{P}), \quad \sigma_{3i}(\mathbf{x};\mathscr{P}) = 0, \quad \mathbf{x} \in \mathbb{R}^2 \setminus G(\mathscr{P}). \tag{4}$$

In the problem of vertical indentation an isotropic or transversally isotropic media by an axi-symmetric punch, the contact region is always a circle. This fact simplifies analysis of the problem. The analysis of three-dimensional contact is usually more complicated.

For the general case of the problem of vertical pressing, we have

$$g(\mathbf{x};\mathscr{P}) = h - f(x_1, x_2). \tag{5}$$

If one considers the frictionless problem, then the following two conditions hold within the contact region

$$\sigma_{31}(\mathbf{x};\mathscr{P}) = \sigma_{32}(\mathbf{x};\mathscr{P}) = 0, \quad \mathbf{x} \in G(\mathscr{P}) \subset \mathbb{R}^2. \tag{6}$$

The conditions within the contact region for adhesive contact will be considered later.

2.3. The Galin solution and the BASh relation

The BASh relation was originally derived for spherical and conical indenters using exact solutions for these indenters collected in Lur'e's book (Lur'e, 1955). However, the relation could be easily obtained from the Galin solution for arbitrary indenter of a monomial shape that he obtained in 1946 (Galin, 1946) (see also Galin, 1953). Indeed, applying his general solution to the case of axi-symmetric punches whose shape is described by monomial functions

$$f(\rho) = B_d \rho^d, \tag{7}$$

Galin (1946) derived the following formulae (see, also Galin, 1953, p. 162, Eqs. (5.34) and (5.35))

$$P = \frac{E}{1-\nu^2} B_d \frac{d^2}{d+1} 2^{d-1} \frac{[\Gamma(d/2)]^2}{\Gamma(d)} a^{d+1}, \quad h = B_d d 2^{d-2} \frac{[\Gamma(d/2)]^2}{\Gamma(d)} a^d. \tag{8}$$

Using (8), he established the following relation between the force P and the displacement h

$$P = \frac{E}{1-\nu^2} \left[B_d^{-1/d} 2^{2/d} d^{(d-1)/d} \frac{1}{d+1} [\Gamma(d/2)]^{-2/d} [\Gamma(d)]^{1/d} \right] h^{(d+1)/d}. \tag{9}$$

Here d is the degree of the monomial function, and $\Gamma(d)$ is the Euler gamma function. Note (9) was presented by Galin (1946, 1953) (his Eqs. (4.35) and (5.36) respectively) with an omitted sign minus at the exponent of B_d.

Differentiating (9) and using the second part of equations (8), one can see that

$$\frac{\mathrm{d}P}{\mathrm{d}h} = E^* \left[\frac{4}{B_d d} \frac{\Gamma(d)}{[\Gamma(d/2)]^2} \right]^{1/d} h^{1/d} = 2E^* a. \tag{10}$$

This is in accordance with the BASh relation (1) because $a = \sqrt{A/\pi}$.

Further, taking into account that the shape function for bodies of revolution may be presented in the form of the power series with fractional exponents

$$f(\rho) = \sum_{k=1}^{\infty} B_k \rho^{d_k}, \quad d_k > 0 \tag{11}$$

and that Hertz type contact problems with identical contact regions can be superimposed on each other (see, e.g. Mossakovskii, 1963; Borodich, 1990), one could prove the validity of the BASh relation (1) for any blunt axi-symmetric indenter. However, this result was obtained by Pharr et al. (1992) using another approach.

2.4. The Galin solution for frictionless axi-symmetric contact

Various analytical approaches were developed to extract mechanical properties of materials from indentation load–displacement data. The approaches are mainly based on either Galin's or Sneddon's formulae. As we noted, both of these formulae are valid only for axi-symmetric frictionless Hertz-type contact problems.

Let us use Cartesian and cylindrical coordinate frames, namely $x_1 = x$, $x_2 = y$, $x_3 = z$ and ρ, ϕ, z, where $\rho = \sqrt{x^2 + y^2}$ and $x = \rho\cos\phi$, $y = \rho\sin\phi$.

In 1946 considering axi-symmetric frictionless contact problems for an elastic isotropic half-space, Galin obtained expressions for the contacting force P, the depth of penetration h and the pressure distribution under a convex, smooth in $\mathbb{R}^2 \setminus \{0\}$ punch of the arbitrary shape $x_3 = -f(\rho)$, $f(0) = 0$. In particular, he wrote (see Eqs. (5.29) and (5.30) by Galin, 1946) or Eqs. (5.29) and (5.30) by Galin (1953)

$$P = \frac{2E}{1-\nu^2}\int_0^a \rho_1 \Delta f(\rho_1)\sqrt{a^2 - \rho_1^2}\,\mathrm{d}\rho_1, \tag{12}$$

$$h = \int_0^a \rho_1 \Delta f(\rho_1)\mathrm{arctanh}\left(\sqrt{1 - \rho_1^2/a^2}\right)\mathrm{d}\rho_1. \tag{13}$$

Here a is the radius of contact and Δ denotes the two-dimensional Laplace operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}. \tag{14}$$

If the shape function is described by (7) then $\Delta f(\rho) = Bd^2\rho^{d-2}$, and (12) and (13) lead to (8). The solution for an indenter described by (11) can be obtained as a superposition of solutions to the Hertz type contact problems having the same fixed contact radius a. Hence, the contact load P and the depth of indentation h satisfy the following equations (Borodich, 1990)

$$P = E^* \sum_{k=1}^{\infty} A(B_k, d_k)a^{d_k+1}, \quad A(B_k, d_k) = B_k 2^{d_k-1}\frac{d_k^2}{d_k+1}\frac{[\Gamma(d_k/2)]^2}{\Gamma(d_k)},$$
$$h = \sum_{k=1}^{\infty} B_k d_k 2^{d_k-2}\frac{[\Gamma(d_k/2)]^2}{\Gamma(d_k)}a^{d_k}. \tag{15}$$

(Note Borodich (1990) omitted the coefficient 1/2 in (15)) Differentiating P and h in with respect to a, one obtains the BASh relation (1).

Let us show that the BASh relation (1) can be derived directly from the Galin solution, namely (12)–(14). To do this we employ the Leibnitz rule of differentiation of an integral by a parameter α

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_{L_1(\alpha)}^{L_2(\alpha)} F(x,\alpha)\,\mathrm{d}x = \int_{L_1(\alpha)}^{L_2(\alpha)} \frac{\mathrm{d}F(x,\alpha)}{\mathrm{d}\alpha}\mathrm{d}x + F(L_2,\alpha)\frac{\mathrm{d}L_2}{\mathrm{d}\alpha} - F(L_1,\alpha)\frac{\mathrm{d}L_1}{\mathrm{d}\alpha}.$$

For both Eqs. (12) and (13), the parameter $\alpha = a$, the limits of integrations $L_1 = 0$ and $L_2 = a$, while $F(L_2, \alpha) = 0$. Hence, we have

$$\frac{\mathrm{d}P}{\mathrm{d}a} = 2E^*\int_0^a \rho_1 \Delta f(\rho_1)\frac{\mathrm{d}\sqrt{a^2-\rho_1^2}}{\mathrm{d}a}\mathrm{d}\rho_1 = 2E^* a\int_0^a \rho_1 \Delta f(\rho_1)\frac{1}{\sqrt{a^2-\rho_1^2}}\mathrm{d}\rho_1, \tag{16}$$

$$\frac{\mathrm{d}h}{\mathrm{d}a} = \int_0^a \rho_1 \Delta f(\rho_1) \frac{\mathrm{d}\left[\operatorname{arctanh}\left(\sqrt{1-\rho_1^2/a^2}\right)\right]}{\mathrm{d}a}\,\mathrm{d}\rho_1. \tag{17}$$

Taking into account the definition (see, e.g., (4.6.3) and (4.6.22) in Abramovitz and Stegun (1964))

$$\operatorname{arctanh} v = \int_0^v \frac{\mathrm{d}t}{1-t^2} = \frac{1}{2}\ln\frac{1+v}{1-v} \tag{18}$$

and substituting $v = \sqrt{1-\rho_1^2/a^2}$ into this formula, one obtains

$$\frac{\mathrm{d}[\operatorname{arctanh} v]}{\mathrm{d}a} = \frac{1}{1-v^2}\frac{\rho_1^2 a^{-3}}{\sqrt{1-\rho_1^2/a^2}} = \frac{1}{\sqrt{a^2-\rho_1^2}}. \tag{19}$$

By substituting this formula into (17) and comparing the result with (16), we obtain the BASh relation for frictionless contact

$$\frac{\mathrm{d}P}{\mathrm{d}h} = \frac{\mathrm{d}P/\mathrm{d}a}{\mathrm{d}h/\mathrm{d}a} = 2aE^* = \frac{2\sqrt{A}}{\sqrt{\pi}}E^*.$$

Remark 1. Sometimes the derivation of the BASh relation is wrongly attributed to Sneddon (1965). In fact, Sneddon (1965) presented the formulae for the force and the indentation depth of a punch having contact radius $\rho = a$, namely

$$P = \frac{4\mu a}{1-v}\int_0^1 \frac{\xi^2 w'(\xi)\,\mathrm{d}\xi}{\sqrt{1-\xi^2}}, \tag{20}$$

$$h = \int_0^1 \frac{w'(\xi)\,\mathrm{d}\xi}{\sqrt{1-\xi^2}}, \tag{21}$$

where $f(\rho) = w(\rho/a)$ and $\mu = E/(2(1+v))$. Although Sneddon derived independently the formulae (20) and (21), he could have obtained them from the Galin (1946) solution (see Appendix A). Similarly to the above derivation, the BASh relation can be also derived the Sneddon formulae (see Appendix B). However, neither Galin nor Sneddon calculated the slope $\mathrm{d}P/\mathrm{d}h$. The BASh relation was presented only in 1975 (Bulychev et al., 1975).

Remark 2. Using the property of the Euler gamma functions $\Gamma(n+1) = n!$, it is possible to show that the Shtaerman (1939) solution is a particular case $d = 2n$ of the Galin solution (9). Here n is a natural number. In particular, one can obtain the Shtaerman (1939) formula (see also Eq (5.20) in Johnson's (1985) book)

$$P = 4nB\frac{E}{1-v^2}a^{2n+1}\frac{2\cdot 4\cdots 2n}{1\cdot 3\cdots 2n+1} = 4nB\frac{E}{1-v^2}a^{2n+1}\frac{(2n)!!}{(2n+1)!!}. \tag{22}$$

Remark 3. It follows from (8) that

$$P = \frac{2d}{d+1}\cdot\frac{E}{1-v^2}ah(a). \tag{23}$$

In particular, taking a limit $d \to \infty$ in (8), one obtains the Boussinesq relation for a flat ended cylindrical indenter of the radius a

$$P = \frac{2E}{1-\nu^2} a h(a). \tag{24}$$

3. Adhesive (no-slip) axi-symmetric contact problems

Let us consider next the adhesive axi-symmetric Hertz type contact problems. If the external parameter of the problem $\mathscr{P}$ is gradually increased then the surface displacements $u_r(r, 0, \mathscr{P})$ and $u_z(r, 0, \mathscr{P})$ will be functions of both r and the parameter of the problem $\mathscr{P}$. Once the point of the surface contacts with the indenter, its radial displacement does not change further with $\mathscr{P}$. Hence, instead of the conditions (6), one can write the following no-slip condition within the contact region

$$\frac{\partial u_r}{\partial \mathscr{P}}(r, 0, \mathscr{P}) = 0. \tag{25}$$

The analysis of the adhesive contact problems was performed first incrementally (Mossakovskii, 1954, 1963; Goodman, 1962) for a growth in the contact radius a. Mossakovskii noted self-similarity of the problem for punches described by monomial functions (7). However, only Spence (1968) pointed out that the solution can be obtained directly without application of the incremental techniques (see, Johnson, 1985; Gladwell, 1980). Self-similarity of a general frictional Hertz type contact problem was shown later by Borodich (1993).

Mossakovskii (1954, 1963) considered only two particular examples of no-slip contact problems, namely the problems for a flat-ended cylinder and a parabolic punch. Spence (1968) introduced an alternative method for solution of the problems, corrected some misprints in the Mosskovskii examples and presented also the solution to the problem for a conical punch.

Following Mossakovskii and Spence, let us take the contact radius a as the external parameter of the problem $\mathscr{P}$.

3.1. The Mossakovskii solution for adhesive contact

In 1954 Mossakovskii presented the solution to a mixed boundary value problem for an elastic half-space when the line separating the boundary conditions is a circle. As an example, he gave a solution for a flat-ended circular punch of the radius a under condition of adhesive (no-slip) contact. Mossakovskii presented the following formula for the compressing normal stresses σ_{zz} under a flat-ended punch of the radius a under condition of adhesive (no-slip) contact

$$\sigma_{zz}^0(\rho, 0, a) = K h_0 \frac{1}{\rho} \frac{\mathrm{d}}{\mathrm{d}\rho} \int_0^\rho \sin\left(\beta \ln \frac{a-x}{a+x}\right) \frac{x}{\sqrt{\rho^2 - x^2}} \, \mathrm{d}x. \tag{26}$$

Here h_0 is the depth of the punch and

$$\beta = \frac{1}{2\pi} \ln(3 - 4\nu), \quad K = \frac{8\mu(1-\nu)}{\pi(1-2\nu)\sqrt{3-4\nu}}.$$

The correctness of the formula (26) was later checked by Keer (1967) and Spence (1968). Speaking about the further calculations of the compressing stress by Mossakovskii, Spence (1968) made a remark that a factor of 2 was omitted throughout his paper of 1963, beginning with his equation (2.16). Indeed, Mossakovskii's papers have various misprints; for example, Mossakovskii's expression for the contact force obtained by integration of the pressure (26) over the contact region, should be (see Spence, 1968, and Khadem and Keer, 1974)

$$P = 4\mu h_0 a \frac{\ln(3-4\nu)}{1-2\nu}.$$

However, at this instance his calculations were correct and Spence was in error. The above formula was also presented with a misprint in Johnson's book (1985), see (3.105). One can see that the solution differs from the frictionless Boussinesq solution (24).

Integrating (26) by parts, one obtains the following formula for the pressure under a circular plane punch with unit settlement

$$\sigma^0_{zz}(\rho, 0, a) = -2\beta a K \int_0^\rho \frac{\chi(x,a)\,\mathrm{d}x}{\sqrt{\rho^2 - x^2(a^2-x^2)}}, \quad \chi(x,a) = \cos\left(\beta \ln\frac{a-x}{a+x}\right). \tag{27}$$

Applying the incremental approach to the solution (27) with varying radius t of the punch, one can calculate the normal stress under a curved axi-symmetric punch

$$\sigma_{zz}(\rho, 0, a) = \int_\rho^a h'(t)\sigma^0_{zz}(\rho, 0, t)\,\mathrm{d}t. \tag{28}$$

Developing the Mossakovskii approach, Borodich and Keer (2004) obtained the following formula for the contact force

$$P(a) = \frac{16\mu(1-\nu)\ln(3-4\nu)}{\pi(1-2\nu)\sqrt{3-4\nu}} I \int_0^a h'(t)t\,\mathrm{d}t, \quad I = \int_0^a \frac{\chi(x,a)}{\sqrt{a^2-x^2}}\,\mathrm{d}x. \tag{29}$$

The integral I can be calculated using the Abramov–Muskhelishvili solution to the two-dimensional problem of adhesive contact between a punch with straight horizontal base and an elastic half-plane (see Appendix C)

$$I = \frac{\pi}{4}\frac{\sqrt{3-4\nu}}{1-\nu}.$$

Hence, it follows from (29) that the general expression for the force acting on a curved axi-symmetric punch at adhesive contact, is

$$P(a) = \frac{4\mu\ln(3-4\nu)}{(1-2\nu)} \int_0^a h'(t)t\,\mathrm{d}t. \tag{30}$$

By differentiating (30) with respect to a, one obtains that the slope of the P–h curve is

$$\frac{\mathrm{d}P}{\mathrm{d}h} = \frac{P'(a)}{h'(a)} = \frac{4\mu\ln(3-4\nu)}{(1-2\nu)} a \tag{31}$$

or

$$S = \frac{\mathrm{d}P}{\mathrm{d}h} = C\frac{2E}{1-\nu^2}\frac{\sqrt{A}}{\sqrt{\pi}}. \tag{32}$$

Thus, the BASh relation (1) should be corrected by the factor C in the case of frictional contact, where in the case of adhesive (no-slip) contact

$$C = \frac{(1-\nu)\ln(3-4\nu)}{1-2\nu}. \tag{33}$$

This factor decreases from $C = \ln 3 = 1.0986$ at $\nu = 0$ and takes its minimum $C = 1$ at $\nu = 0.5$. Taking into account that full adhesion preventing any slip within the contact region is not the case for real physical contact and there is some frictional slip at the edge of the contact region (see Galin 1945, 1953; Spence, 1975), we can conclude that the values of the correction factor C in (32) cannot exceed the upper bound (33).

3.2. Solution to the problem for punches of monomial shape

Let us consider in detail the adhesive contact for punches of monomial shape. In the adhesive contact problem, the equation for the determination of the derivative of the sought function $h'(t)$ of displacements under the punch of the shape $x_3 = -f(\rho)$ has the form (Mossakovskii, 1963)

$$f(\rho) = \frac{2}{\pi}\int_0^{\rho} \frac{1}{\sqrt{\rho^2 - x^2}}\left[\int_0^x h'(t)\cos\left(\beta\ln\frac{x-t}{x+t}\right)\mathrm{d}t\right]\mathrm{d}x. \tag{34}$$

It follows from (34) that if $h'(t) = K_d t^{d-1}$ or $h(t) = K_d t^d/d$ then $f(\rho) = B_d\rho^d$ where

$$B_d = K_d C_d, \quad C_d = \frac{2}{\pi} I^*(d) I^{**}(d) \tag{35}$$

and

$$I^*(d) = \int_0^1 t^{d-1}\cos\left(\beta\ln\frac{1-t}{1+t}\right)\mathrm{d}t, \quad I^{**}(d) = \int_0^1 \frac{x^d}{\sqrt{1-x^2}}\mathrm{d}x.$$

Taking into account

$$I^{**}(d) = \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+2}{2}\right)} = \frac{1}{d}\frac{\sqrt{\pi}\,\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} = \frac{2^{1-d}\pi}{d}\frac{\Gamma(d)}{\left[\Gamma\left(\frac{d}{2}\right)\right]^2},$$

one obtains

$$C_d = \frac{2^{2-d}}{d}\frac{\Gamma(d)}{\left[\Gamma\left(\frac{d}{2}\right)\right]^2} I^*(d).$$

It follows from (30) that the force is

$$P(a) = \frac{4\mu\ln(3-4\nu)}{(1-2\nu)}\cdot\frac{B_d}{C_d}\cdot\frac{a^{d+1}}{d+1}. \tag{36}$$

Thus, in the case of axi-symmetric punches whose shape is described by monomial functions (7), the relations between the force P and the contact radius a and between the displacement h and a are given by the following exact formulae

$$P = \frac{E\ln(3-4\nu)}{(1+\nu)(1-2\nu)} B_d \frac{d}{d+1} 2^{d-1}\frac{[\Gamma(d/2)]^2}{\Gamma(d)}\frac{1}{I^*(d)} a^{d+1}, \quad h = B_d 2^{d-2}\frac{[\Gamma(d/2)]^2}{\Gamma(d)}\frac{1}{I^*(d)} a^d. \tag{37}$$

Using (37), one can establish the following relation between the force P and the displacement h for a monomial punch in the case of adhesive contact

$$P = \frac{E\ln(3-4\nu)}{(1+\nu)(1-2\nu)}\frac{d}{d+1}\left[\frac{4I^*(d)}{B_d}\frac{\Gamma(d)}{[\Gamma(d/2)]^2}\right]^{1/d} h^{\frac{d+1}{d}}. \tag{38}$$

In the case $\nu = 0.5$, one has

$$\lim_{\nu\to 0.5}\frac{E\ln(3-4\nu)}{(1+\nu)(1-2\nu)}=\frac{4E}{3},$$

$\beta=0$ and $I^*(d)=1/d$. Hence, the formulae (37) and (38) are identical with the corresponding formulae (8) and (9) obtained by Galin (1946) for frictionless contact.

Using the above general solution for monomial punches, we can consider some particular cases.

Conical punch. In the case of a cone of semi-vertical angle $\pi/2-\alpha$, $d=1$, $f(\rho)=B_1\rho$, and $h'(a)=K_1$. For a linearized treatment to be possible, α must be small compared with 1 and $\tan\alpha=B_1\approx\alpha$. It follows from (36) that the force is

$$P=\frac{2\mu\ln(3-4\nu)}{1-2\nu}\frac{B_1}{C_1}a^2.$$

Taking into account that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$ and $\Gamma(1)=1$, one obtains from (37)

$$P=\frac{\pi\mu\ln(3-4\nu)}{(1-2\nu)I^*(1)}B_1a^2. \tag{39}$$

$I^*(1)$ can be represented as the following Fourier transform (see (4.6) by Spence (1968))

$$I^*(1)=\int_0^1\cos\left(\beta\ln\frac{1-t}{1+t}\right)\mathrm{d}t=\int_0^\infty\cos(\beta\xi/2)\mathrm{sech}^2\xi\,\mathrm{d}\xi,\quad \xi(t)=\frac{1}{2}\ln\frac{1+t}{1-t}$$

and using tables collected by Erdelyi (1954, p. 30)

$$I^*(1)=\pi\beta\mathrm{cosech}(\pi\beta)=\frac{2\pi\beta}{(e^{\pi\beta}-e^{-\pi\beta})}=\frac{\ln(3-4\nu)\sqrt{3-4\nu}}{2(1-2\nu)}. \tag{40}$$

Substituting (40) into (39), we obtain

$$P=\frac{2\mu\pi B_1a^2}{\sqrt{3-4\nu}}.$$

The adhesive problem for a cone was first considered by Spence (1968). The above relation is the same as obtained by Spence (1968) (see his equation (4.26)).

Spherical punch. In the case of a sphere of radius R, $d=2$, $B_2=1/(2R)$, $f(\rho)=B_2\rho^2$, and

$$h'(a)=K_2a=a/(2RC_2). \tag{41}$$

It follows from (37)

$$P=\frac{2\mu\ln(3-4\nu)}{3R(1-2\nu)}\frac{a^3}{C_2}=\frac{4\mu\ln(3-4\nu)}{3R(1-2\nu)I^*(2)}a^3.$$

The adhesive problem for a sphere was first considered by Mossakovskii (1963) and Spence (1968). Our constant C_2 is d_1 in Mossakovskii's notation and $\gamma(\kappa)/4$ in Spence's notation. Their results are identical with the above, except for a factor 2 which was omitted by Mossakovskii in his equation (5.6) (this is because he omitted this factor earlier in his equation (5.2) which is our (41)) and factor $\gamma(\kappa)$ which was omitted by Spence in his equation (4.20).

4. Similarity considerations of 3D indentation

So far, we have considered axi-symmetric indenters and isotropic, linear elastic materials. However, if the indenter is neither a sphere nor a cone, but is either a Vickers or Berkovich indenter whose tip is a nominally four-sided or a three-sided pyramid, respectively, then the axi-symmetric solutions are not valid. Also, the solution is not valid when the tested material is anisotropic. In addition, real indenters have some deviation from their nominal shapes. Hence, it is important to derive theoretical formulae, which are valid for general 3D schemes of nanoindentation by indenters of non-ideal shapes.

The conditions under which frictionless Hertz type contact problems possess classical self-similarity, are as follows (Borodich, 1988): *the constitutive relationships are homogeneous with respect to the strains or the stresses and the indenter's shape is described by a homogeneous function whose degree is greater than or equal to unity. It is also assumed that during the process of the contact, the loading at any point is progressive.* This mean that the functions of the indenter's shape f should satisfy the identity $f(\lambda x_1, \lambda x_2) = \lambda^d f(x_1, x_2)$, for arbitrary positive λ. Here d is the degree of the homogeneous function f, in particular, $d = 2$ for the elliptic paraboloid considered by Hertz. Additionally, operators of constitutive relations $\mathscr{F}$ for materials of contacting bodies should be homogeneous functions of degree κ with respect to the components of the strain tensor e_{ij}, i.e.,

$$\mathscr{F}(\lambda e_{ij}) = \lambda^\kappa \mathscr{F}(e_{ij}). \tag{42}$$

The theoretical analysis of Hertz-type contact problems based on similarity transformations of the 3D contact problems does not depend on the anisotropy of the material (Borodich, 1990, 1993). The material behavior of the medium may be linear or non-linear, anisotropic or isotropic, depending on the form of the operator $\mathscr{F}$. Hooke's law is an example of the linear ($\kappa = 1$) homogeneous constitutive relationships. Another example is the constitutive relationships of a plastic isotropic non-compressible material of the form

$$\sigma_{ij}^D = K\Gamma^{\kappa-1}\epsilon_{ij}, \tag{43}$$

where δ_{ij} is the Kroneker delta, σ_{ij}^D are components of stress deviator,

$$\sigma_{ij}^D = \sigma_{ij} - \delta_{ij}\sigma, \quad \sigma = \sigma_{ii}/3, \quad \Gamma = \sqrt{\epsilon_{ij}^D \epsilon_{ij}^D/2},$$

where Γ is the intensity of shear strains, and K and κ are material constants. The constitutive relationships of a plastic anisotropic materials given by Pobedrya (1984) are also homogeneous. These relationships are often noted as the power law of material hardening.

Let P_1 be some initial value of the external load, $l(P_1)$ and $h(P_1)$ be respectively the characteristic size of the contact region and the depth of indentation (displacement) at this load. Then l and h at any other value of the load for monomial indenters and materials with power-law stress–strain relations can be re-scaled using the following formulae (Borodich, 1989, 1993):

$$\begin{aligned} h(c,P) &= c^{(2-\kappa)/[2+\kappa(d-1)]}(P/P_I)^{d/[2+\kappa(d-1)]}h(1,P_I), \\ l(c,P) &= c^{-\kappa/[2+\kappa(d-1)]}(P/P_I)^{1/[2+\kappa(d-1)]}l(1,P_I). \end{aligned} \tag{44}$$

For linear elastic materials, $\kappa = 1$. Hence, one has $h \sim P^{d/(d+1)}$ and $l \sim P^{1/(d+1)}$. This is in accordance with Galin's (1946) formulae (8) and (9) for isotropic materials and in the case of $d = 2$ with the Willis' (1966) solution for anisotropic elastic solids.

We have considered above the adhesive contact conditions. Let us denote the quantities referring to the body $x_3^+ \leqslant 0$ by a superscript "plus" sign, and those referring to the second body by a superscript "minus" sign. In the adhesive contact problem, there is no relative slip between the bodies within the contact region. If the following values are introduced

$$v_1(x_1,x_2) \equiv u_1^+(x_1,x_2,0,P) - u_1^-(x_1,x_2,0,P)$$

and

$$v_2(x_1,x_2) \equiv u_2^+(x_1,x_2,0,P) - u_2^-(x_1,x_2,0,P),$$

then the condition within this region is that these values do not change with augmentation of the external parameter $\mathscr{P}$. These conditions can be expressed by

$$\frac{\partial}{\partial \mathscr{P}} v_i(x_1,x_2,0,\mathscr{P}) = 0, \quad \mathrm{d}\mathscr{P} > 0. \tag{45}$$

In the frictional contact problem, it is usually assumed (see Bryant and Keer, 1982) that the contact region consists of the following parts: in the inner part G_1 the interfacial friction must be sufficient to prevent any slip taking place between bodies, i.e., (42) holds; in the outer part $G \setminus G_1$ the friction must satisfy the Coulomb frictional law (Vermeulen and Johnson, 1964; Spektor, 1981). Let us define the vector of tangential stresses $\boldsymbol{\tau}^{\pm}(x_1,x_2,0,P) \equiv (\sigma_{31}^{\pm}(x_1,x_2,0,P), \sigma_{32}^{\pm}(x_1,x_2,0,P))$. Then the frictional contact conditions can be written as

$$\frac{\partial}{\partial \mathscr{P}} v_i(x_1,x_2,0,\mathscr{P}) = 0, \quad \mathrm{d}\mathscr{P} > 0, \quad (x_1,x_2) \in G_1,$$

$$\boldsymbol{\tau}^{\pm}(x_1,x_2,0,P) = -\theta\sigma_{33}^{\pm}(x_1,x_2,0,P)\left[\frac{\mathbf{v}(x_1,x_2,0,P)}{|\mathbf{v}(x_1,x_2,0,P)|}\right], \quad (x_1,x_2) \in G \setminus G_1. \tag{46}$$

The re-scaling formulae (44) are valid not only in the case of frictionless contact but also for frictional contact problems, in particular when both regions of stick and slip are within the contact region (see the above conditions (46)).

Let us denote by P_1, A_1, l_1 and h_1 respectively some initial load, the corresponding contact area, the characteristic size of the contact region and the displacement. Then (44) can be re-written as

$$\frac{l}{l_1} = c^{\frac{-\kappa}{2+\kappa(d-1)}}\left(\frac{P}{P_1}\right)^{\frac{1}{2+\kappa(d-1)}}, \quad \frac{h}{h_1} = c^{\frac{2-\kappa}{2+\kappa(d-1)}}\left(\frac{P}{P_1}\right)^{\frac{d}{2+\kappa(d-1)}} \tag{47}$$

and as shown by Borodich et al. (2003), the re-scaling formula for the contact area is

$$\frac{A}{A_1} = c^{-2/d}\left(\frac{h}{h_1}\right)^{2/d}. \tag{48}$$

If one considers the same indenter then $c = 1$. It follows from (48) that if the indenter tip is described as a monomial function of degree d, then $h \sim A^{d/2}$ independently of the work hardening exponent κ.

For a fixed indenter, i.e. $c = 1$, the hardness is the following function of the depth of indentation

$$\frac{H}{H_1} = \left(\frac{h}{h_1}\right)^{\frac{\kappa(d-1)}{d}}.$$

However, for an ideal conical or pyramid-shaped indenters $d = 1$ and the hardness is constant.

The above re-scaling formulae (47) and (48) were obtained assuming the homogeneity of material properties and that the stress–strain relation remains the same for any depth of indentation. This is not always true (see a review by Ioffe, 1949). In addition, it is known that plastic deformation exhibits a strong dependence on size below micrometer length scales (see, e.g. Gao et al., 1999 and literature therein). One possible way to model these effects is to employ models of strain gradient plasticity. However, as we have seen above, non-ideal indenter geometries can also affect the interpretation of the experimental results.

Using (48), one can calibrate the indenter tip from area–displacement curve. An example of such a curve was given by Doerner and Nix (1986). Employing (48), one can obtain from their data that the indenter shape for $h \leqslant 90$ nm can be described as a monomial function of degree $d = 1.44$ (see Fig. 2).

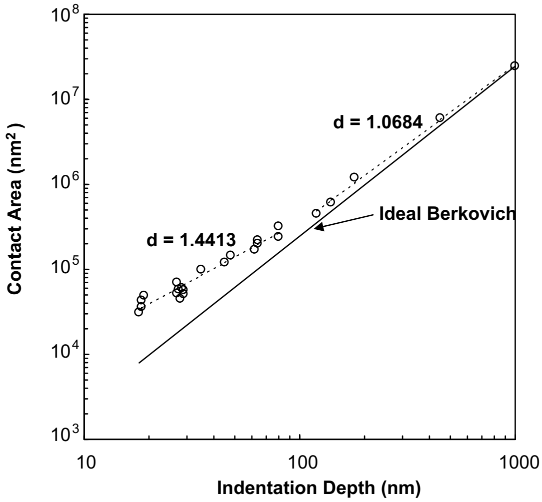

Fig. 2. The area–indentation depth curve for a non-ideal Berkovich indenter (after Borodich et al., 2003). Experimental data from Doerner and Nix (1986).

5. Conclusion

Nanoindentation techniques provide a unique opportunity to obtain mechanical properties of materials of very small volumes. There is also a correlation between the tensile stress–strain curve and the hardness (see, e.g. Ludwik, 1927; Davidenkov, 1943; Zaitsev, 1949; Menčik, 1996; and Dao et al., 2001). Various plasticity characteristic can be also obtained through hardness measurements (Milman et al., 1993). The load–displacement and load–area curves are the basis for nanoindentation tests, and their interpretation is usually based on the main assumptions of the Hertz contact theory and formulae obtained for ideally shaped indenters. We have re-examined some fundamental relations of the nanoindentation mechanics and studied the influence of frictional boundary conditions on the relations.

We have showed that the BASh relation (1) can be derived for frictionless contact using various ways: directly from the Galin (1946) solution, from representation of the shape function in the form of series, and directly from the Sneddon (1965) solution. For frictional contact, the formula for the stiffness is given by (32), i.e. the BASh relation should be corrected by a factor C. The upper value of the factor is given by (33) obtained for adhesive contact.

Then we have concentrated on indenters of monomial shape and derived exact solutions for adhesive axi-symmetric contact. The obtained formulae (37) and (38) coincide with Galin's frictionless formulae (8) and (9) when the material is incompressible.

For monomial indenters and homogeneous constitutive relations, contact problems obey the self-similar laws. Using similarity considerations of 3D contact problems (Borodich, 1989, 1993) and the corresponding formulae, the fundamental relations (47) and (48) are derived for depth of indentation, size of the contact region, load, hardness, and contact area, which are valid for both linear and non-linear, isotropic and anisotropic materials. For loading, the formulae depend on the material hardening exponent κ and the degree of the monomial function of the shape d, in particular $h \sim A^{d/2}$ independently of the work hardening exponent.

It is widely accepted that the most significant source of uncertainty in nanoindentation measurement is the deviation of the indenter tip from nominal geometry (Herrmann et al., 2000). Hence, the formulae (37) and (38) are especially important for shallow indentation (usually less than 100 nm) where the tip bluntness is on the same order as the indentation depth. It follows from our studies that some uncertainties in nanoindentation measurements, which are sometimes attributed to properties of the material, can be explained and quantitatively described by properly accounting for geometric deviation of the indenter tip from its nominal geometry.

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Appendix A. The Sneddon representation of the Galin solution

Let us derive formulae (20) and (21) from the Galin solution (12) and (13). Substituting (14) into (12), one has

$$\frac{P}{2E^*} = I_1 + I_2, \quad I_1 = \int_0^a f''(\rho_1)\rho_1\sqrt{a^2-\rho_1^2}\,\mathrm{d}\rho_1, \quad I_2 = \int_0^a f'(\rho_1)\sqrt{a^2-\rho_1^2}\,\mathrm{d}\rho_1.$$

Integrating by parts, one obtains

$$I_1 = \left[f'(\rho_1)\rho_1\sqrt{a^2-\rho_1^2}\right]_0^a - \int_0^a f'(\rho_1)d\left(\rho_1\sqrt{a^2-\rho_1^2}\right) = -I_2 + \int_0^a \frac{\rho_1^2 f'(\rho_1)\,\mathrm{d}\rho_1}{\sqrt{a^2-\rho_1^2}}.$$

Hence,

$$\frac{P}{2E^*} = \int_0^a \frac{\rho_1^2 f'(\rho_1)\,\mathrm{d}\rho_1}{\sqrt{a^2-\rho_1^2}} = \int_0^a \frac{\rho_1^2\,\mathrm{d}f(\rho_1)}{\sqrt{a^2-\rho_1^2}}. \tag{A.1}$$

By making a substitution $\xi = \rho_1/a$ and using shear modulus μ instead of E, one obtains (20)

$$\frac{1-\nu}{4\mu}P = a\int_0^1 \frac{\xi^2\,\mathrm{d}w(\xi)}{\sqrt{1-\xi^2}}.$$

Similarly, substituting (14) into (13), one has $h = I_3 + I_4$, where

$$I_3 = \int_0^a \rho_1 f''(\rho_1)\mathrm{arctanh}\left(\sqrt{1-\rho_1^2/a^2}\right)\mathrm{d}\rho_1, \quad I_4 = \int_0^a f'(\rho_1)\mathrm{arctanh}\left(\sqrt{1-\rho_1^2/a^2}\right)\mathrm{d}\rho_1.$$

Integrating by parts, one obtains

$$I_3 = \left[f'(\rho_1)\rho_1 \text{arctanh}\left(\sqrt{1-\rho_1^2/a^2}\right)\right]_0^a - \int_0^a f'(\rho_1)\mathrm{d}\left(\rho_1 \text{arctanh}\left(\sqrt{1-\rho_1^2/a^2}\right)\right)$$
$$= -I_4 - \int_0^a \rho_1 f'(\rho_1)\mathrm{d}\left(\text{arctanh}\left(\sqrt{1-\rho_1^2/a^2}\right)\right).$$

Hence, one has

$$h = -\int_0^a \rho_1 f'(\rho_1)\mathrm{d}\left[\text{arctanh}\left(\sqrt{1-\rho_1^2/a^2}\right)\right].$$

Taking into account (18), one obtains the following representation of Galin's formula

$$h(a) = \int_0^a \frac{f'(\rho_1)}{\sqrt{1-\rho_1^2/a^2}}\,\mathrm{d}\rho_1. \tag{A.2}$$

As above, a substitution $\xi = \rho_1/a$ leads to the Sneddon formula for the depth of indentation of a axi-symmetric punch having contact radius $\rho = a$

$$h = \int_0^a \frac{\mathrm{d}f(\rho_1)}{\sqrt{1-\rho_1^2/a^2}} = \int_0^1 \frac{\mathrm{d}w(\xi)}{\sqrt{1-\xi^2}}.$$

Appendix B. The BASh relation for frictionless axi-symmetric punches

Let us derive the BASh relation from the Sneddon representation. This way of derivation is similar to the way used by Pharr et al. (1992). It follows from (A.1) and (A.2)

$$P\frac{1-\nu^2}{2E} = a^2\int_0^a \frac{f'(\rho_1)\,\mathrm{d}\rho_1}{\sqrt{a^2-\rho_1^2}} - \int_0^a \sqrt{a^2-\rho_1^2} f'(\rho_1)\,\mathrm{d}\rho_1 = ah(a) - \int_0^a \sqrt{a^2-\rho_1^2} f'(\rho_1)\,\mathrm{d}\rho_1. \tag{B.1}$$

Differentiating (B.1) by a, one has

$$\frac{1-\nu^2}{2E}\frac{\mathrm{d}P}{\mathrm{d}a} = h(a) + a\frac{\mathrm{d}h(a)}{\mathrm{d}a} - \frac{\mathrm{d}}{\mathrm{d}a}\int_0^a \sqrt{a^2-\rho_1^2} f'(\rho_1)\,\mathrm{d}\rho_1.$$

However,

$$\frac{\mathrm{d}}{\mathrm{d}a}\int_0^a \sqrt{a^2-\rho_1^2} f'(\rho_1)\,\mathrm{d}\rho_1 = \sqrt{a^2-\rho_1^2} f'(\rho_1)\Big|_{\rho_1=a} + \int_0^a \frac{\mathrm{d}}{\mathrm{d}a}\sqrt{a^2-\rho_1^2} f'(\rho_1)\,\mathrm{d}\rho_1 = h(a).$$

or

$$\frac{1-\nu^2}{2E}\frac{\mathrm{d}P}{\mathrm{d}a} = h(a) + a\frac{\mathrm{d}h(a)}{\mathrm{d}a} - h(a) = a\frac{\mathrm{d}h(a)}{\mathrm{d}a}.$$

Hence, one obtains (1)

$$\frac{1-\nu^2}{2E}\frac{\mathrm{d}P}{\mathrm{d}h} = \frac{1-\nu^2}{2E}\frac{\frac{\mathrm{d}P}{\mathrm{d}a}}{\frac{\mathrm{d}h(a)}{\mathrm{d}a}} = a.$$

Appendix C. Two-dimensional problem for a punch with horizontal base

First, an effective solution to the two-dimensional problem of adhesive contact between a punch with straight horizontal base and an elastic half-plane was given by Abramov (1937) using Mellin's integrals. Then Muskhelishvili (1949) gave another solution to the problem using Kolosov's (1914) complex potentials. The pressure $p(x)$ under the punch $-a \leqslant x \leqslant a$ loaded by a vertical force P_0 is determined by the following formula

$$p(x) = \frac{2P_0(1-\nu)}{\pi\sqrt{a^2-x^2}\sqrt{3-4\nu}}\cos\left[\beta\ln\frac{a+x}{a-x}\right].$$

(Note the coefficient 1/2 omitted by Muskhelishvili (1949) in his equations (114.7a) and (114.8a).)

On the other hand, one has

$$P_0 = \int_{-a}^{a} p(x)\,\mathrm{d}x = 2\int_0^a p(x)\,\mathrm{d}x = \frac{4P_0(1-\nu)}{\pi\sqrt{3-4\nu}}\int_0^a \frac{\chi(x,a)}{\sqrt{a^2-x^2}}\,\mathrm{d}x.$$

Hence,

$$I = \int_0^a \frac{\chi(x,a)}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \frac{\pi}{4}\frac{\sqrt{3-4\nu}}{1-\nu}.$$

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